Connected Operators for the Totally Asymmetric Exclusion Process

O. Golinelli, K. Mallick

Service de Physique Théorique, Cea Saclay, 91191 Gif, France

6 April 2007

Abstract

We fully elucidate the structure of the hierarchy of the connected operators that commute with the Markov matrix of the Totally Asymmetric Exclusion Process (TASEP). We prove for the connected operators a combinatorial formula that was conjectured in a previous work. Our derivation is purely algebraic and relies on the algebra generated by the local jump operators involved in the TASEP.

Keywords: Non-Equilibrium Statistical Mechanics, ASEP, Exact Results, Algebraic Bethe Ansatz.

Pacs numbers: 02.30.Ik, 02.50.-r, 75.10.Pq.

1 Introduction

The Asymmetric Simple Exclusion Process (ASEP) is a lattice model of particles with hard core interactions. Due to its simplicity, the ASEP appears as a minimal model in many different contexts such as one-dimensional transport phenomena, molecular motors and traffic models. From a theoretical point of view, this model has become a paradigm in the field of non-equilibrium statistical mechanics; many exact results have been derived using various methods, such as continuous limits, Bethe Ansatz and matrix Ansatz (for reviews, see e.g., Spohn 1991, Derrida 1998, Schütz 2001, Golinelli and Mallick 2006).

In a recent work (Golinelli and Mallick 2007), we applied the algebraic Bethe Ansatz technique to the Totally Asymmetric Exclusion Process (TASEP).

This method allowed us to construct a hierarchy of 'generalized Hamiltonians' that contain the Markov matrix and commute with each other. Using the algebraic relations satisfied by the local jump operators, we derived explicit formulae for the transfer matrix and the generalized Hamiltonians, generated from the transfer matrix. We showed that the transfer matrix can be interpreted as the generator of a discrete time Markov process and we described the actions of the generalized Hamiltonians. These actions are non-local because they involve non-connected bonds of the lattice. However, connected operators are generated by taking the logarithm of the transfer matrix. We conjectured for the connected operators a combinatorial formula that was verified for the first ten connected operators by using a symbolic calculation program.

The aim of the present work is to present an analytical calculation of the connected operators and to prove the formula that was proposed in (Golinelli and Mallick 2007). This paper is a sequel of our previous work, however, in section 2, we briefly review the main definitions and results already obtained so that this work can be read in a fairly self-contained manner. In section 3, we derive the general expression of the connected operators.

2 Review of known results

We first recall the dynamical rules that define the TASEP with n particles on a periodic 1-d ring with L sites labelled $i = 1, \ldots, L$. The particles move according to the following dynamics: during the time interval $[t, t + dt]$, a particle on a site i jumps with probability dt to the neighboring site $i + 1$, if this site is empty. This *exclusion rule* which forbids to have more than one particle per site, mimics a hard-core interaction between particles. Because the particles can jump only in one direction this process is called totally asymmetric. The total number n of particles is conserved. The TASEP being a continuous-time Markov process, its dynamics is entirely encoded in a $2^L \times 2^L$ Markov matrix M, that describes the evolution of the probability distribution of the system at time t. The Markov matrix can be written as

$$M = \sum_{i=1}^{L} M_i \,, \tag{1}$$

where the local jump operator M_i affects only the sites i and $i + 1$ and represents the contribution to the dynamics of jumps from the site i to $i+1$.

2.1 The TASEP algebra

The local jump operators satisfy a set of algebraic equations :

$$\begin{aligned} M_i^2 &= -M_i, && (2)\\ M_i\ M_{i+1}\ M_i = M_{i+1}\ M_i\ M_{i+1} &= 0, && (3)\\ [M_i, M_j] &= 0 \quad \text{if } |i-j| > 1. && (4) \end{aligned}$$

These relations can be obtained as a limiting form of the Temperley-Lieb algebra. On the ring we have periodic boundary conditions : $M_{i+L} = M_i$. The local jumps matrices define an algebra. Any product of the M_i's will be called a *word.* The *length* of a given word is the minimal number of operators M_i required to write it. A word, that can not be simplified further by using the algebraic rules above, will be called a reduced word.

Consider any word W and call $\mathcal{I}(W)$ the set of indices i of the operators M_i that compose it (indices are enumerated without repetitions). We remark that, if W is not annihilated by application of rule (3), the simplification rules (2, 4) do not alter the set $\mathcal{I}(W)$, *i.e.,* these rules do not introduce any new index or suppress any existing index in $\mathcal{I}(W)$. This crucial property is not valid for the algebra associated with the partially asymmetric exclusion process (see Golinelli and Mallick 2006).

Using the relation (2) we observe that for any i and any real number $\lambda \neq 1$ we have

$$(1+\lambda M_i)^{-1} = (1+\alpha M_i) \quad \text{with} \quad \alpha = \frac{\lambda}{\lambda - 1}\,. \tag{5}$$

2.2 Simple words

A simple word of length k is defined as a word $M_{\sigma(1)}M_{\sigma(2)}\ldots M_{\sigma(k)}$, where σ is a permutation on the set $\{1,2,\ldots,k\}$. The commutation rule (4) implies that only the relative position of M_i with respect to $M_{i\pm 1}$ matters. A simple word of length k can therefore be written as $W_k(s_2, s_3, \ldots, s_k)$ where the boolean variable s_j for $2 \leq j \leq k$ is defined as follows : $s_j = 0$ if M_j is on the left of M_{j-1} and $s_j = 1$ if M_j is on the right of M_{j-1}. Equivalently, $W_k(s_2, s_3, \ldots, s_k)$ is uniquely defined by the recursion relation

$$\begin{aligned} W_k(s_2, s_3, \ldots, s_{k-1}, 1) &= W_{k-1}(s_2, s_3, \ldots, s_{k-1})\ M_k\ , && (6)\\ W_k(s_2, s_3, \ldots, s_{k-1}, 0) &= M_k\ W_{k-1}(s_2, s_3, \ldots, s_{k-1})\,. && (7) \end{aligned}$$

The set of the 2^{k-1}simple words of length k will be called $\mathcal{W}_k$. For a simple word W_k, we define $u(W_k)$ to be the number of *inversions* in W_k, *i.e.*, the

number of times that M_j is on the left of M_{j-1} :

$$u(W_k(s_2, s_3, \ldots, s_k)) = \sum_{j=2}^{k} (1 - s_j) \ . \tag{8}$$

We remark that simple words are connected, they cannot be factorized in two (or more) commuting words.

2.3 Ring-ordered product

Because of the periodic boundary conditions, products of local jump operators must be ordered adequately. In the following we shall need to use a ring ordered product $\mathcal{O}\left(\right)$ which acts on words of the type

$$W = M_{i_1} M_{i_2} \ldots M_{i_k} \ \text{ with } \ 1 \leq i_1 < i_2 < \ldots < i_k \leq L \,, \tag{9}$$

by changing the positions of matrices that appear in W according to the following rules :

(i) If $i_1 > 1$ or $i_k < L$, we define $\mathcal{O}\left(W\right) = W$. The word W is well-ordered.

(ii) If $i_1 = 1$ and $i_k = L$, we first write W as a product of two blocks, $W = AB$, such that $B = M_b M_{b+1} \ldots M_L$ is the maximal block of matrices with consecutive indices that contains M_L, and $A = M_1 M_{i_2} \ldots M_{i_a}$, with $i_a < b - 1$, contains the remaining terms. We then define

$$\mathcal{O}\left(W\right) = \mathcal{O}\left(AB\right) = BA = M_b M_{b+1} \ldots M_L M_1 M_{i_2} \ldots M_{i_a} \,. \tag{10}$$

(iii) The previous definition makes sense only for $k < L$. Indeed, when $k = L$, we have $W = M_1 M_2 \ldots M_L$ and it is not possible to split W in two different blocks A and B. For this special case, we define

$$\mathcal{O}\left(M_1 M_2 \ldots M_L\right) = |1, 1, \ldots, 1\rangle\langle 1, 1, \ldots, 1| \,, \tag{11}$$

which is the projector on the 'full' configuration with all sites occupied.

The ring-ordering $\mathcal{O}\left(\right)$ is extended by linearity to the vector space spanned by words of the type described above.

2.4 Transfer matrix and generalized Hamiltonians H_k

The algebraic Bethe Ansatz allows to construct a one parameter commuting family of transfer matrices, $t(\lambda)$, that contains the translation operator $T = t(1)$ and the Markov matrix $M = t'(0)$. For $0 \leq \lambda \leq 1$, the operator

$t(\lambda)$ can be interpreted as a discrete time process with non-local jumps : a hole located on the right of a cluster of p particles can jump a distance k in the backward direction, with probability $\lambda^k(1-\lambda)$ for $1 \leq k < p$, and with probability λ^p for $k = p$. The probability that this hole does not jump at all is $1-\lambda$. This model is equivalent to the 3-D anisotropic percolation model of Rajesh and Dhar (1998) and to a 2-D five-vertex model. It is also an adaptation on a periodic lattice of the ASEP with a backward-ordered sequential update (Rajewsky et al. 1996, Brankov et al. 2004), and equivalently of an asymmetric fragmentation process (Rákos and Schütz 2005).

The operator $t(\lambda)$ is a polynomial in λ of degree L given by

$$t(\lambda) = 1 + \sum_{k=1}^{L} \lambda^k H_k \ , \tag{12}$$

where the generalized Hamiltonians H_k are *non-local operators* that act on the configuration space. [We emphasize that the notation used here is different from that of our previous work : $t(\lambda)$ was denoted by $t_g(\lambda)$ in (Golinelli and Mallick 2007).]

We have $H_1 = M$ and more generally, as shown in (Golinelli and Mallick 2007), H_k is a homogeneous sum of words of length k

$$H_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq L} \mathcal{O}\left(M_{i_1} M_{i_2} \ldots M_{i_k}\right) \ , \tag{13}$$

where $\mathcal{O}\,()$ represents the ring ordered product that embodies the periodicity and the translation-invariance constraints.

For a system of size L with N particles only $H_1, H_2, \ldots, H_N$ have a non-trivial action. Because we are interested only in the case $N \leq L-1$ (the full system as no dynamics) there are at most $L-1$ operators H_k that have a non-trivial action.

3 The connected operators F_k

3.1 Definition

The generalized Hamiltonians H_k and the transfer matrix $t(\lambda)$ have non-local actions and couple particles with arbitrary distances between them. Besides H_k is a highly non-extensive quantity as it involves generically a number of terms of order L^k. As usual, the local connected and extensive operators are obtained by taking the logarithm of the transfer matrix. For $k \geq 1$, the

connected Hamiltonians F_k are defined as

$$\ln t(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda^k}{k} F_k \ . \tag{14}$$

Taking the derivative of this equation with respect to λ and recalling that $t(\lambda)$ commutes with $t'(\lambda)$, we obtain

$$\sum_{k=1}^{\infty} \lambda^k F_k = \lambda \ t(\lambda)^{-1} \ t'(\lambda) \, . \tag{15}$$

Expanding $t(\lambda)^{-1}$ with respect to λ, this formula allows to calculate F_k as a polynomial function of $H_1, \ldots, H_k$. For example $F_1 = H_1$, $F_2 = 2H_2 - H_1^2$, etc... (see Golinelli and Mallick 2007). By using (13), we observe that F_k is *a priori* a linear combination of products of k local operators M_i. However this expression can be simplified by using the algebraic rules (2, 3, 4) and *in fine*, F_k will be a linear combination of reduced words of length $j \leq k$.

Because of the ring-ordered product that appears in the expression (13) of the H_k's, it is difficult to derive an expression of F_k in terms of the local jump operators. An exact formula for the F_k with $k \leq 10$ was obtained in (Golinelli and Mallick 2007) by using a computer program and a general expression was conjectured for all k. In the following, the conjectured formula is derived and proved rigorously.

3.2 Elimination of the ring-ordered product

The expression $\sum \lambda^k F_k$ can be written as a linear combination of reduced words W. We know from formula (13) that at most $L-1$ operators H_k are independent in a system of size L, we shall therefore calculate F_k only for $k \leq L-1$. Thus, we need to consider reduced words of length $j \leq L-1$. Let W be such a word, and $\mathcal{I}(W)$ be the set of indices of the operators M_i that compose W; our aim is to find the expression of W and to calculate its prefactor from equation (15). Because the rules (2, 4) do not suppress or add any new index, the following property is true : if a word W' appearing in $\lambda \ t(\lambda)^{-1} \ t'(\lambda)$ is such that $\mathcal{I}(W') \neq \mathcal{I}(W)$ then even after simplification, W' will remain different from W. Therefore, the prefactor of W in $\sum \lambda^k F_k$ is the same as the prefactor of W in

$$\lambda \ t_{\mathcal{I}}(\lambda)^{-1} \ t'_{\mathcal{I}}(\lambda) \quad \text{where } t_{\mathcal{I}}(\lambda) = \mathcal{O}\left(\prod_{i \in \mathcal{I}} (1 + \lambda M_i) \right) \quad \text{with } \mathcal{I}(W) \subset \mathcal{I} \, . \tag{16}$$

Because F_k commutes with the translation operator T, then for any $r = 1, \ldots, L-1$, the prefactor of $W = M_{i_1} M_{i_2} \ldots M_{i_j}$ is the same as the prefactor

of $T^r M T^{-r} = M_{r+i_1} M_{r+i_2} \dots M_{r+i_j}$. Furthermore, any word W of size $k \leq L-1$ is equivalent, by a translation, to a word that contains M_1 and not M_L : indeed, there exists at least one index i_0 such that $i_0 \notin \mathcal{I}(W)$ and $(i_0+1) \in \mathcal{I}(W)$ and it is thus sufficient to translate W by $r = L - i_0$.

In conclusion, it is enough to study in expression (15), the reduced words W with set of indices included in

$$\mathcal{I}^* = \{1, 2, \dots, L-1\}\,. \tag{17}$$

Because the index L does not appear in $\mathcal{I}^*$, the ring-ordered product has a trivial action in equation (16) and we have

$$t_{\mathcal{I}^*}(\lambda) = (1+\lambda M_1)(1+\lambda M_2)\dots(1+\lambda M_{L-1})\,. \tag{18}$$

We have thus been able to eliminate the ring-ordered product.

3.3 Explicit formula for the connected operators

In equation (18), differentiating $t_{\mathcal{I}^*}(\lambda)$ with respect to λ, we have

$$t'_{\mathcal{I}^*}(\lambda) = \sum_{i=1}^{L-1} (1+\lambda M_1)\dots(1+\lambda M_{i-1}) M_i (1+\lambda M_{i+1})\dots(1+\lambda M_{L-1})\,. \tag{19}$$

Using equation (5) we obtain

$$t_{\mathcal{I}^*}(\lambda)^{-1} = (1+\alpha M_{L-1})(1+\alpha M_{L-2})\dots(1+\alpha M_1)\,, \text{ with } \alpha = \frac{\lambda}{\lambda-1}\,. \tag{20}$$

Noticing that $\lambda(1+\alpha M_i)M_i = -\alpha M_i$, we deduce

$$\begin{aligned}
&\lambda\; t_{\mathcal{I}^*}(\lambda)^{-1}\; t'_{\mathcal{I}^*}(\lambda) = \qquad\qquad\qquad\qquad (21)\\
&-\alpha \sum_{i=1}^{L-1} (1+\alpha M_{L-1})\dots(1+\alpha M_{i+1}) M_i (1+\lambda M_{i+1})\dots(1+\lambda M_{L-1})\,.
\end{aligned}$$

The ith term in this sum contains words with indices between i and $L-1$. Because we are looking for the words that contain the operator M_1, we must consider only the first term in this sum, which we note by Q

$$Q = -\alpha(1+\alpha M_{L-1})\dots(1+\alpha M_2) M_1 (1+\lambda M_2)\dots(1+\lambda M_{L-1})\,. \tag{22}$$

In the appendix, we show that

$$Q = R_1 + R_2 + \ldots + R_{L-1}\,, \tag{23}$$

where R_i is defined by the recursion :

$$
\begin{aligned}
R_1 &= -\alpha M_1 \, , && (24)\\
R_i &= \lambda R_{i-1} M_i + \alpha M_i R_{i-1} \quad \text{for } i \geq 2\, . && (25)
\end{aligned}
$$

To summarize, all the words in $\sum_{k=1}^{\infty} \lambda^k F_k$ that contain M_1 and not M_L are given by $Q = R_1 + R_2 + \ldots + R_{L-1}$. From the recursion relation (25) we deduce that R_i is a linear combination of the 2^{i-1} simple words $W_i(s_2, s_3, \ldots, s_i)$ defined in section 2.1. Furthermore, we observe from (25) that a factor λ appears if $s_i = 1$ and a factor $\alpha = \lambda/(\lambda - 1)$ appears if $s_i = 0$. Therefore, the coefficient $f(W)$ of $W = W_i(s_2, s_3, \ldots, s_i)$ in Q is given by

$$
f(W) = (-1)^u \frac{\lambda^i}{(1-\lambda)^{u+1}} = (-1)^u \sum_{j=0}^{\infty} \binom{u+j}{j} \lambda^{i+j} \tag{26}
$$

where i is the length of W and $u = u(W)$ is its inversion number, defined in equation (8). We have thus shown that

$$
Q = \sum_{i=1}^{L-1} \sum_{W \in \mathcal{W}_i} f(W)\; W = \sum_{i=1}^{L-1} \sum_{W \in \mathcal{W}_i} W \sum_{j=0}^{\infty} (-1)^{u(W)} \binom{u(W)+j}{j} \lambda^{i+j} \, , \tag{27}
$$

where $\mathcal{W}_i$ is the set of simple words of length i.

Finally, we recall that the coefficient in $\sum_{k=1}^{\infty} \lambda^k F_k$ of a reduced word W that contains M_1 and not M_L is the same as its coefficient in Q. Extracting the term of order λ^k in equation (27) we deduce that any word W in F_k that contains M_1 and not M_L is a simple word of length $i \leq k$ and its prefactor is given by $(-1)^{u(W)} \binom{u(W)+k-i}{k-i}$.

The full expression of F_k is obtained by applying the translation operator to the expression (27); indeed any word in F_k can be uniquely obtained by translating a simple word in F_k that contains M_1 and not M_L. We conclude that for $k < L$,

$$
F_k = \mathcal{T} \sum_{i=1}^{k} \sum_{W \in \mathcal{W}_i} (-1)^{u(W)} \binom{k-i+u(W)}{k-i} W \, , \tag{28}
$$

where $\mathcal{T}$ is the translation-symmetrizator that acts on any operator A as follows : $\mathcal{T} A = \sum_{i=0}^{L-1} T^i \; A \; T^{-i}$. The presence of $\mathcal{T}$ in equation (28) insures that F_k is invariant by translation on the periodic system of size L. All simple words being connected, we finally remark that formula (28) implies that F_k is a connected operator.

4 Conclusion

By using the algebraic properties of the TASEP algebra (2-4), we have derived an exact combinatorial expression for the family of connected operators that commute with the Markov matrix. This calculation allows to fully elucidate the hierarchical structure obtained from the Algebraic Bethe Ansatz. It would be of a great interest to extend our result to the partially asymmetric exclusion process (PASEP), in which a particle can make forward jumps with probability p and backward jumps with probability q. In particular, we recall that the symmetric exclusion process is equivalent to the Heisenberg spin chain : in this case the connected operators have been calculated only for the lowest orders (Fabricius et al., 1990). This is a challenging and difficult problem. In our derivation we used a fundamental property of the TASEP algebra : the rules (2-4) when applied to a word W either cancel W or conserve the set of indices $\mathcal{I}(W)$. The algebra associated with PASEP violates this crucial property because there we have $M_i\ M_{i+1}\ M_i = pq\ M_i$. Therefore the method followed here does not have a straightforward extension to the PASEP case.

Appendix: Proof of equation (23)

Let us define the following series

$$\begin{aligned} Q_1 &= -\alpha M_1\ , && (29)\\ Q_i &= (1+\alpha M_i)Q_{i-1}(1+\lambda M_i) \quad \text{for } i \geq 2\,. && (30)\end{aligned}$$

We remark that Q defined in equation (22) is given by $Q = Q_{L-1}$. Let us consider R_i defined by the recursion (25). The indices that appear in the words of Q_i and R_i belong to $\{1, 2, \ldots, i\}$. Therefore, we have

$$[R_j, M_i] = 0 \quad \text{for} \quad j \leq i-2\,, \tag{31}$$

because the operators $M_1, M_2, \ldots, M_j$ that compose R_j commute with M_i. From equations (31) and (5), we obtain

$$(1+\alpha M_i)R_j(1+\lambda M_i) = R_j \quad \text{for } j \leq i-2\,. \tag{32}$$

Furthermore, from (25), we obtain

$$M_i R_{i-1} M_i = \lambda M_i R_{i-2} M_{i-1} M_i + \alpha M_i M_{i-1} R_{i-2} M_i\,. \tag{33}$$

Because M_i commutes with R_{i-2}, we can use the relation $M_i M_{i-1} M_i = 0$ to deduce that

$$M_i R_{i-1} M_i = 0\,. \tag{34}$$

Using equation (34), we find

$$(1 + \alpha M_i)R_{i-1}(1 + \lambda M_i) = R_{i-1} + \lambda R_{i-1}M_i + \alpha M_i R_{i-1} = R_{i-1} + R_i\,. \quad (35)$$

From equations (32) and (35), we prove that the (unique) solution of the recursion relation (30) is given by equation (23), $Q_i = R_1 + R_2 + \ldots + R_i$.

References

- Brankov J. G., Priezzhev V. B. and Shelest R. V., 2004, *Generalized determinant solution of the discrete-time totally asymmetric exclusion process and zero-range process*, Phys. Rev. E **69** 066136.

- Derrida B., 1998, *An exactly soluble non-equilibrium system: the asymmetric simple exclusion process*, Phys. Rep. **301** 65.

- Fabricius K., Mütter K.-H. and Grosse H., 1990, *Hidden symmetries in the one-dimensional antiferromagnetic Heisenberg model*, Phys. Rev. B **42** 4656.

- Golinelli O. and Mallick K., 2006, *The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics,* J. Phys. A: Math. Gen. **39** 12679.

- Golinelli O. and Mallick K., 2007, *Family of Commuting Operators for the Totally Asymmetric Exclusion Process,* Submitted to J. Phys. A: Math. Theor., cond-mat/0612351.

- Rajesh R. and Dhar D., 1998, *An exactly solvable anisotropic directed percolation model in three dimensions*, Phys. Rev. Lett. **81** 1646.

- Rajewsky N., Schadschneider A. and Schreckenberg M., 1996, *The asymmetric exclusion model with sequential update*, J. Phys. A: Math. Gen. **29** L305.

- Rákos A. and Schütz G. M., 2005, *Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process*, J. Stat. Phys. **118** 511.

- Schütz G. M., 2001, *Exactly solvable models for many-body systems far from equilibrium* in *Phase Transitions and Critical Phenomena,* vol. 19, C. Domb and J. L. Lebowitz Ed., Academic Press, San Diego.

- Spohn H., 1991, *Large scale dynamics of interacting particles*, Springer, New-York.